B.Sc. Semester-II Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 22111 **Course Code : SH/MTH/201/C-3**

Course Title : Real Analysis

[NEW SYLLABUS]

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

UNIT-I

1. Answer any **five** from the following questions:

2×5=10

a) Let A and B be two nonempty bounded subsets of $\mathbb{R}$. Prove that

$\inf(A \cup B) = min\{\inf A, \inf B\}$.

b) Find sup A and inf A, where

$A = \left\{\frac{n+(-1)^n}{n} : n \in \mathbb{N}\right\}$.

c) Prove that a convergent sequence is a cauchy sequence.

d) If $0 < a < b$ determine $\lim_{n\to\infty}\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$.

e) Find the limit of the sequence $\{x_n\}$, where $x_1 = \sqrt{6}$ and $x_{n+1} = \sqrt{6+x_n}$ for $n \geq 1$.

f) Show that the series $\sum_{n=1}^{\infty}\frac{1}{n+n^2}$ converges.

g) Examine if the set

$S = \{x \in \mathbb{R} : 2x^2 - 5x + 2 > 0\}$ is open in $\mathbb{R}$.

h) Let S be a bounded subset of $\mathbb{R}$ and T be a non-empty subset of S. Prove that $\sup T \leq \sup S$.

UNIT-II

2. Answer any **four** from the following questions:

5×4=20

a) i) Let $\{x_n\}$ be a bounded sequence. Then show that $\overline{lim}x_n = \underline{lim}x_n$. Is $\{x_n\}$ convergent? Justify.

ii) Let $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Then show that

$\underline{lim}x_n + \underline{lim}y_n \leq \underline{lim}(x_n + y_n)$.

b) i) Let S be a non-empty subset of $\mathbb{R}$ that is bouded above, and let a be any number in $\mathbb{R}$. Define the set

$a + S := \{a + s : s \in S\}$. Prove that $\sup(a + S) = a + \sup(S)$.

ii) Give an example of a divergent sequence $\{a_n\}$ of positive numbers with $\lim_{n\to\infty}\left\{(a_n)^{\frac{1}{n}}\right\} = 1$. 3+2

c) Let $x_1 = 8$ and $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that $\{x_n\}$ is bounded and monotone. Find the limit.

d) i) Test the convergence of the series $1 - \frac{1}{2}\left(1 + \frac{1}{3}\right) + \frac{1}{3}\left(1 + \frac{1}{3} + \frac{1}{5}\right) - \ldots\ldots\ldots$

ii) Show that the series $1 - \frac{1}{4} + \frac{1}{7} - \frac{1}{10} + \cdots$ is conditionally convergent. 3+2

e) i) If $x \in \mathbb{R}$ and $x > 0$, show that there exists a natural number m such that $m - 1 \leq x < m$.

ii) Show that an infinite subset of an enumerable set is enumerable. 3+2

f) i) Define interior point of a subset of $\mathbb{R}$. Let S be a subset of $\mathbb{R}$, prove that an interior point of S is a limit point of S.

ii) Give an example of subset of $\mathbb{R}$ which is neither open nor cosed in $\mathbb{R}$. 4+1

UNIT-III

3. Answer any **one** of the following questions: 10×1=10

a) i) Give an example of unbounded sequence that have a convergent subsequence.

ii) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$ is absolutely or conditionally convergent.

iii) Find the derive set of A where $A = \left\{\frac{1}{2^m} + \frac{1}{2^n} : m, n \in \mathbb{N}\right\}$. 2+5+3

b) i) If $\lim_{n\to\infty} x_n = l$ then show that $\lim \frac{x_1 + x_2 + \cdots + x_n}{n} = l$.

ii) Let $u_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$ and $v_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \log n; n \geq 2$. Show that $\{u_n\}_{n=1}^{\infty}$ is a monotone decreasing sequence and $\{v_n\}_{n=2}^{\infty}$ is a monotone increasing one and they converge to the same limit.

iii) Using the definition of a compact set, prove that a finite subset of $\mathbb{R}$ is a compact set in $\mathbb{R}$.

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